Isometric-path numbers of block graphs

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Revised on March 10, 2004

Abstract

An isometric path between two vertices in a graph G is a shortest path joining them. The isometric-path number of G, denoted by $\mathrm{ip}(G)$, is the minimum number of isometric paths required to cover all vertices of G. In this paper, we determine exact values of isometric-path numbers of block graphs. We also give a linear-time algorithm for finding the corresponding paths.

Keywords. Isometric path, block graph, cut-vertex, algorithm

1 Introduction

An *isometric path* between two vertices in a graph G is a shortest path joining them. The *isometric-path number* of G, denoted by $\mathrm{ip}(G)$, is the minimum number of isometric paths required to cover all vertices of G. This concept has a close relationship with the game of cops and robbers described as follows. The game is played by two players, the *cop* and the *robber*, on a graph. The two players move alternatively, starting with the cop. Each player's first move consists of choosing a vertex at which to start. At each subsequent move, a player may choose either to stay at the same vertex or to move to an adjacent

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vertex. The object for the cop is to catch the robber, and for the robber is to prevent this from happening. Nowakowski and Winkler [7] and Quilliot [8] independently proved that the cop wins if and only if the graph can be reduced to a single vertex by successively removing pitfalls, where a *pitfall* is a vertex whose closed neighborhood is a subset of the closed neighborhood of another vertex. As not all graphs are cop-win graphs, Aigner and Fromme [1] introduced the concept of the *cop-number* of a general graph G, denoted by $c(G)$, which is the minimum number of cops needed to put into the graph in order to catch the robber . On the way to giving an upper bound for the cop-numbers of planar graphs, they showed that a single cop moving on an isometric path P guarantee that after a finite number of moves the robber will be immediately caught if he moves onto P. Observing this fact, Fitzpatrick [4] then introduced the concept of isometric-path cover and pointed out that $c(G) \leq \mathrm{ip}(G)$.

The isometric-path number of the Cartesian product $P_{n_1} \times P_{n_2} \times \ldots \times P_{n_d}$ has been studied in the literature. Fitzpatrick [5] gave bounds for the case when $n_1 = n_2 = \ldots = n_d$. Fisher and Fitzpatrick [3] gave exact values for the case $d = 2$. Fitzpatrick et al [6] gave a lower bound, which is in fact the exact value if $d + 1$ is a power of 2, for the case when $n_1 = n_2 = \ldots = n_d = 2$.

The purpose of this paper is to give exact values of isometric-path numbers of block graphs. We also give a linear-time algorithm to find the corresponding paths. For technical reasons, we consider a slightly more general problem as follows. Suppose every vertex v in the graph G is associated with a non-negative integer $f(v)$. We call such function f a *vertex labeling* of G. An f-*isometric-path cover* of G is a family $\mathcal{C}$ of isometric paths such that the following conditions hold.

(C1) If $f(v) = 0$, then v is in an isometric path in $\mathcal{C}$.

(C2) If $f(v) \geq 1$, then v is an end vertex of at least $f(v)$ isometric paths in $\mathcal{C}$, while the counting is twice if v itself is a path in $\mathcal{C}$.

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The *f-isometric-path number* of G, denoted by $\mathrm{ip}_f(G)$, is the minimum cardinality of an f-isometric-path cover of G. It is clear that when $f(v) = 0$ for all vertices v in G, we have $\mathrm{ip}(G) = \mathrm{ip}_f(G)$. The attempt of is paper is to determine the f-isometric-path number of a block graph. Recall that a *block graph* is a graph in which every block is a complete graph. A *cut-vertex* of a graph is a vertex whose removal results in a graph with more components than the original graph. It is well-known that in a block graph all internal vertices of an isometric path are cut-vertices.

2   Block graphs

In this section, we determine the f-isometric-path numbers for block graphs G. Without loss of generality, we may assume that G is connected.

First, a useful lemma.

Lemma 1 *Suppose x is a non-cut-vertex of a block graph G with a vertex labeling f. If vertex labeling f' is the same as f except that $f'(x) = \max\{1, f(x)\}$, then $\mathrm{ip}_f(G) = \mathrm{ip}_{f'}(G)$.*

Proof. As any internal vertex of an isometric path in a block graph is a cut-vertex but x not a cut-vertex, x must be an end vertex of any isometric path. It follows that a collection $\mathcal{C}$ is an f-isometric-path cover if and only if it is an f'-isometric-path cover. The lemma then follows. ∎

So, now we may assume that $f(v) \geq 1$ for all non-cut-vertices v of G, and call such a vertex labeling *regular*. Now, we have the following theorem for the inductive step.

Theorem 2 *Suppose G is a block graph with a regular labeling f, and x is a non-cut-vertex in a block B with exactly one cut-vertex y or with no cut-vertex in which case let y be any vertex of $B - \{x\}$. When $f(x) = 1$, let $G' = G - x$ with a regular vertex labeling f' which is the same as f except $f'(y) = f(y) + 1$. When $f(x) \geq 2$, let $G' = G$ with a regular vertex labeling f' which is the same as f except $f'(x) = f(x) - 1$ and $f'(y) = f(y) + 1$. Then $\mathrm{ip}_f(G) = \mathrm{ip}_{f'}(G')$.*

Proof. We first prove that $\mathrm{ip}_f(G) \geq \mathrm{ip}_{f'}(G')$. Suppose $\mathcal{C}$ is an optimal f-isometric-path cover of G. Choose a path P in $\mathcal{C}$ having x as an end vertex. We consider four cases.

Case 1.1. $P = x$ and $f(x) = 1$ (i.e., $G' = G - x$).

In this case, $\mathcal{C}' = (\mathcal{C} - \{P\}) \cup \{y\}$ is an f'-isometric-path cover of G'. Hence, $\mathrm{ip}_f(G) = |\mathcal{C}| \geq |\mathcal{C}'| \geq \mathrm{ip}_{f'}(G')$.

Case 1.2. $P = x$ and $f(x) \geq 2$ (i.e., $G' = G$).

In this case, $\mathcal{C}' = (\mathcal{C} - \{P\}) \cup \{xy\}$ is an f'-isometric-path cover of G'. Hence, $\mathrm{ip}_f(G) = |\mathcal{C}| \geq |\mathcal{C}'| \geq \mathrm{ip}_{f'}(G')$.

Case 1.3. $P = xz$ for some vertex z in $B - \{x, y\}$.

In this case, $\mathcal{C}' = (\mathcal{C} - \{P\}) \cup \{yz\}$ is an f'-isometric-path cover of G'. Hence, $\mathrm{ip}_f(G) = |\mathcal{C}| \geq |\mathcal{C}'| \geq \mathrm{ip}_{f'}(G')$.

Case 1.4. $P = xyQ$, where Q contains no vertices in B.

In this case, $\mathcal{C}' = (\mathcal{C} - \{P\}) \cup \{yQ\}$ is an f'-isometric-path cover of G'. Hence, $\mathrm{ip}_f(G) = |\mathcal{C}| \geq |\mathcal{C}'| \geq \mathrm{ip}_{f'}(G')$.

Next, we prove that $\mathrm{ip}_f(G) \leq \mathrm{ip}_{f'}(G')$. Suppose $\mathcal{C}'$ is an optimal f'-isometric-path cover of G'. Choose a path P' in $\mathcal{C}'$ having y as an end vertex. We consider three cases.

Case 2.1. $P' = yx$.

In this case, $G' = G$ and $\mathcal{C} = (\mathcal{C}' - \{P'\}) \cup \{x\}$ is an f-isometric-path cover of G. Hence, $\mathrm{ip}_f(G) \leq |\mathcal{C}| \leq |\mathcal{C}'| = \mathrm{ip}_{f'}(G')$.

Case 2.2. $P' = yz$ for some z in $B - \{x, y\}$.

In this case, $\mathcal{C} = (\mathcal{C}' - \{P'\}) \cup \{xz\}$ is an f-isometric-path cover of G. Hence, $\mathrm{ip}_f(G) \leq |\mathcal{C}| \leq |\mathcal{C}'| = \mathrm{ip}_{f'}(G')$.

Case 2.3. $P' = yQ$, where Q contains no vertex in B.

In this case, $\mathcal{C} = (\mathcal{C}' - \{P'\}) \cup \{xyQ\}$ is an f-isometric-path cover of G. Hence, $\mathrm{ip}_f(G) \leq |\mathcal{C}| \leq |\mathcal{C}'| = \mathrm{ip}_{f'}(G')$. ∎

Consequently, we have the following result for f-isometric-path numbers of connected block graphs.

Theorem 3 *If G is a connected block graph with a regular vertex labeling f, then $\mathrm{ip}_f(G) = \lceil s(G)/2 \rceil$, where $s(G) = \sum_{v \in V(G)} f(v)$.*

Proof. The theorem is obvious when G has only one vertex. For the case when G has more than one vertex, we apply Theorem 2 repeatedly until the graph becomes trivial. Notice that the $s(G') = s(G)$ when apply Theorem 2. ∎

For the isometric-path-cover problem, we have

Corollary 4 *If G is a connected block graph, then $\mathrm{ip}(G) = \lceil \mathrm{nc}(G)/2 \rceil$, where $\mathrm{nc}(G)$ is the number of non-cut-vertices of G.*

Proof. The corollary follows from Theorem 3 and the fact that $\mathrm{ip}(G) = \mathrm{ip}_f(G)$ for the regular vertex labeling f with $f(v) = 1$ if v is a non-cut-vertex and $f(v) = 0$ otherwise. ∎

3 Algorithm

Based on Theorem 2, we are able to design an algorithm for the isometric-path-cover problem in block graphs. Notice that we may only consider connected block graphs with regular vertex labelings. To speed up the algorithm, we may modify Theorem 2 a little bit so that each time a non-cut-vertex is handled.

Theorem 5 *Suppose G is a block graph with a regular labeling f, and x is a non-cut-vertex in a block B with exactly one cut-vertex y or with no cut-vertex in which let y be any vertex in $B - \{x\}$. Let $G' = G - x$ with a regular vertex labeling f' which is the same as f except $f'(y) = f(y) + f(x)$. Then $\mathrm{ip}_f(G) = \mathrm{ip}_{f'}(G')$.*

Proof. The theorem follows from repeatedly applying Theorem 2. ∎

Now, we are ready to give the algorithm.

Algorithm PG Find the f-isometric-path number $\mathrm{ip}_f(G)$ of a connected block graph.
Input. A connected block graph G and a regular vertex labeling f.

Output. An optimal f-isometric-path cover $\mathcal{C}$ of G and $\mathrm{ip}_f(G)$.

Method.

1.   construct a stack S which is empty at the beginning;

2.   let $G' \leftarrow G$;

3.   **while** (G' has more than one vertex) **do**

4.   choose a block B with exactly one cut-vertex y or with

            no cut-vertex in which case choose any $y \in B$;

5.   **for** (all vertices x in $B - \{y\}$) **do**

6. $f(y) \leftarrow f(y) + f(x)$;

7.   push $(x, y, f(x))$ into S;

8. $G' \leftarrow G' - x$;

9.   **end for**;

10. **end while**;

11. $\mathrm{ip}_f(G) \leftarrow \lceil f(r)/2 \rceil$, where r is the only vertex of G';

12. let $\mathcal{C}$ be the family of isometric paths containing $\mathrm{ip}(G)$ copies of the path r;

13. **while** (S is not empty) **do**

14.   pop (x, y, i) from S;

15.   choose i copies of path P in $\mathcal{C}$ using y as an end vertex;

16.   **if** ($P = yx$) **then**

17.   replace the i copies of P by i copies of x in $\mathcal{C}$;

18.   **if** ($P = yz$ for some vertex z in the block of G containing x) **then**

19.   replace the i copies of P by i copies of xz in $\mathcal{C}$;

20.   **if** ($P = yQ$ where Q has no vertices in the block of G containing x) **then**

21.   replace the i copies of P by the i copies of xyQ in $\mathcal{C}$;

22. **end while**.

Algorithm **PG** can be implemented in time linear to the number of vertices and edges. Notice that we can use the depth-first search to find all blocks and cut-vertices of a graph, see [2].

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